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Dynamics of structurally finite transcendental entire functions

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1 Introduction

We investigate the dynamics of structurally finite transcendental entire functions, which was defined by Taniguchi ([**2**]). We will show that we can define an itinerary for the points which remain in some region under iteration and the set of all points which share the same itinerary forms a curve which goes to infinity. Also these curves belong to the Julia set and the points on these curves tend to infinity under iteration. This is a generalization of the result by Schleicher and Zimmer ([**1**]) for the exponential family.

2 Structurally finite entire functions

In this section, we make a brief explanation of structurally finite entire fucntions which was defined by Taniguchi ([**2**]).

Definition 1 (Maskit surgery by connecting functions)
Let $f_j : \mathbb{C} \to \mathbb{C}$ $(j = 1, 2)$ be two entire functions, and A_j be the set of singular values of f_j.

Assume that there is a cross cut L in $\mathbb{C}$ such that

1. both of $L \cap A_1$ and $L \cap A_2$ are either empty or consist of a single point z_0, which is an isolated point of each A_j,

2. L separates $A_1 \setminus \{z_0\}$ from $A_2 \setminus \{z_0\}$, and

3. if $L \cap A_1 = L \cap A_2 = \{z_0\}$, then $\{z_0\}$ is a critical value of each f_j: for a small disk U with center z_0 such that $U \cap A_j = \{z_0\}$, $f_j^{-1}(U)$ has a relatively compact component W_j which contains a critical point for each f_j.

Then we say that an entire function $f : \mathbb{C} \to \mathbb{C}$ is constructed from f_1 and f_2 by *Maskit surgery* with respect to L if the following assumptions are satisfied: Let D_j be the component of $\mathbb{C} \setminus L$ containing $A_j \setminus \{z_0\}$. Then there exist

1. components $\tilde{D}_1$ and $\tilde{D}_2$ of $f_1^{-1}(D_2)$ and $f_2^{-1}(D_1)$, respectively, such that $f_j : \tilde{D}_j \to D_{3-j}$ is biholomorphic and $\tilde{D}_j \cap W_j \neq \emptyset$ if $L \cap A_j$ are non-empty,
2. a cross cut $\tilde{L}$ in $\mathbb{C}$ such that f gives a homeomorphism of $\tilde{L}$ onto L, and
3. a conformal map ϕ_j of $\mathbb{C} \setminus \tilde{D}_j$ onto U_j such that $f_j = f \circ \phi_j$ on $\mathbb{C} \setminus \tilde{D}_j$, where U_1 and U_2 are components of $\mathbb{C} \setminus \tilde{L}$.

Definition 2 (structurally finite entire functions)
We say that an entire function is *structurally finite* if it can be constructed from a finite number of building blocks by Maskit surgeries. Here, a *building block* is either a *quadratic block*:

$$az^2 + bz + c : \mathbb{C} \to \mathbb{C} \quad (a \neq 0)$$

or an *exponential block*:

$$a \exp(bz) + c : \mathbb{C} \to \mathbb{C} \quad (ab \neq 0).$$

Then the following Representation Theorem holds:

Theorem (Representation Theorem) Every structurally finite entire function has the form

$$\int^z P(t)e^{Q(t)}dt$$

with suitable polynomials P and Q.

Note that if f is structurally finite, then f belongs to so called the Speiser class, which is a class of entire functions with only finite number of singular values.

3 Statement of the result

Let $f(z)$ be a structurally finite transcendental entire function. Since we are interested in the dynamics of f, by the Representation Theorem and some suitable linear conjugation, we can assume that $f(z)$ has the following form:

$$f(z) := a \int_0^z P(t)e^{Q(t)}dt + b,$$

where $a,\ b \in \mathbb{C}$ and $P,\ Q$ are monic polynomials with $\deg P = p \geq 0$, $\deg Q = q \geq 1$. In what follows we consider only transcendental case, we assume here $q \geq 1$. Then it is easy to see that f has p critical values and q asymptotic directions which correspond to some finite asymptotic values. In particular, f has only finite number of singular values. So we take a disk

$$D := \{z \mid |z| < C\}$$

which contains all the singular values of f. Then $f^{-1}(\mathbb{C} \setminus D)$ has exactly q components and each one by one lies in one of the q domains which are divided by the q asymptotic paths

$$e^{\frac{(2k-1)\pi}{q}i}t \quad (t \geq 0), \quad k = 1, 2, \cdots, q,$$

which correspond to some finite asymptotic value. Let Γ be one of these paths, say

$$\Gamma(t) := e^{\frac{\pi}{q}i}t \quad (t \geq 0)$$

for example. Then each connected component of $f^{-1}(\Gamma)$ is a curve which tends to ∞ and its argument tends to one of $\frac{2k\pi}{q}$ $(k = 0, 1, \cdots, q-1)$, which is the argument of asymptotic paths which corresponds to the asymptotic value ∞. Let $\mathcal{S}$ be one of the components of $f^{-1}(\mathbb{C} \setminus D)$. Then we make a partition of $\mathcal{S}$ by using $f^{-1}(\Gamma)$ so that

$$\mathcal{S} = \coprod_{k \in \mathbb{Z}} R_k.$$

(See Figure 1). For a point $z \in \mathcal{S}$ such that $f^n(z) \in \mathcal{S}$ for any $n \in \mathbb{N}$, we can define its itinerary $S(z)$ by

$$S(z) := \underline{s} = s_0 s_1 \cdots s_n \cdots, \quad \text{if } f^n(z) \in R_{s_n}.$$

In what follows, for simplicity, we assume that $\mathcal{S}$ is the component of $f^{-1}(\mathbb{C} \setminus D)$ which has an intersection with $\mathbb{R}^+$.

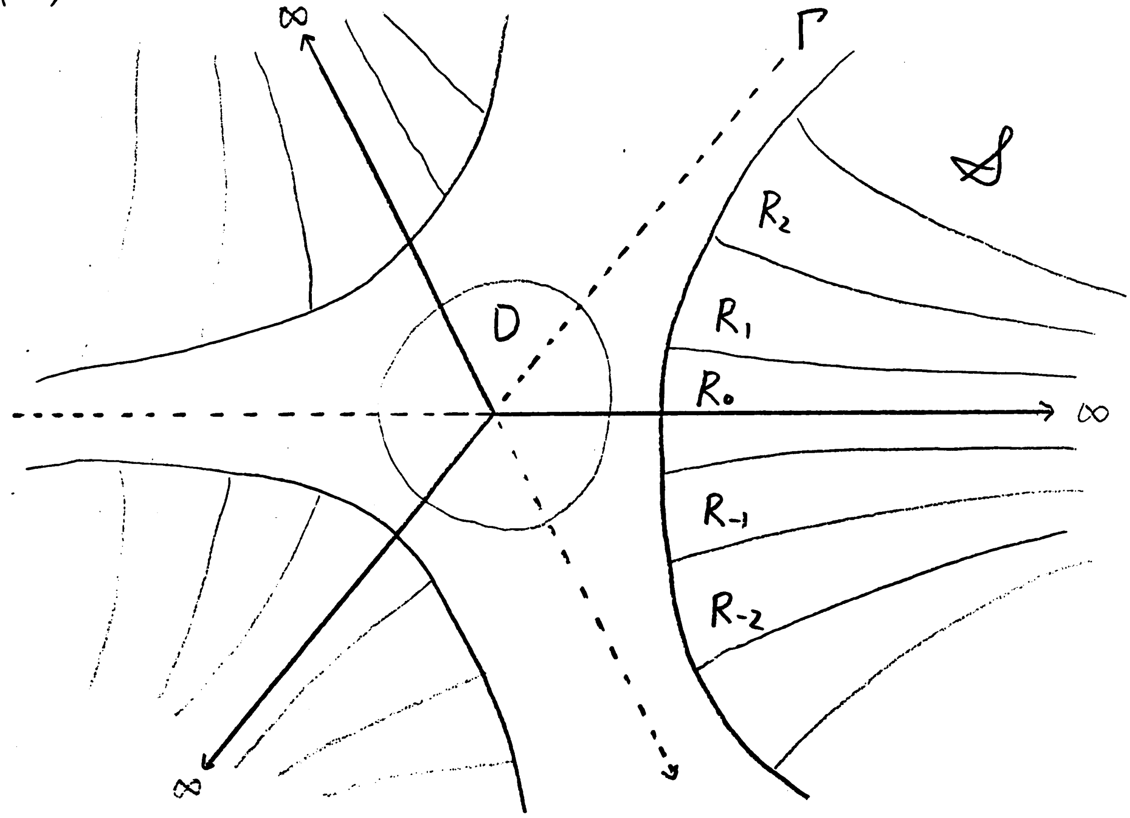

Figure 1 : Domain $\mathcal{S}$ and its partition by $f^{-1}(\Gamma)$ (The case $q = 3$).

Main Theorem Let f be a structurally finite transcendental entire function. For an itinerary $\underline{s} = s_0 s_1 \cdots s_n \cdots$ satisfying $|s_n| \leq F^n(x)$, where $x > 0$ is some constant and

$$F(z) := \sum |a_n| z^n, \quad f(z) = \sum a_n z^n,$$

there exists a continuous curve

$$h_{\underline{s}}(t) \subset \mathcal{S} \quad (t \geq {}^{\exists}t_0)$$

such that

(1) All the points $h_{\underline{s}}(t)$ for fixed t has the itinerary $\underline{s}$.

(2) $f^n(h_{\underline{s}}(t)) \in \mathcal{S}$ for every n.

(3) $f^n(h_{\underline{s}}(t)) \to \infty$ $(n \to \infty)$. In particular, $h_{\underline{s}}(t) \in J(f)$.

(4) $\lim_{t\to\infty} h_{\underline{s}}(t) = \infty$.

(5) $h_{\underline{s}}(t)$ is injective with respect to t.

4 Preliminaries

Proposition 1 If

$$z \in S_0 := \left\{ z \,\middle|\, \left| \arg z - \frac{2k\pi}{q} \right| < \frac{\pi}{4q}, \quad k = 0, \cdots, q-1 \right\}$$

and $|z|$ is sufficiently large, then the following estimates hold:

(1) $f(z) = \frac{a}{q} z^{p-q+1} e^{z^q} (1 + O(|z|^{-1}))$.

(2) $|f(z)| \geq \left|\frac{a}{q}\right| |z|^{p-q+1} \exp(\frac{|z|^q}{\sqrt{2}})$.

(3) $|f(z)| \leq 2 \left|\frac{a}{q}\right| \exp(|z|^{q+\varepsilon})$ for a small $\varepsilon > 0$.

(4) Let g_{s_i} be the branch of f^{-1} which takes values in R_{s_i}. Then

$$|g_{s_i}(z)| \geq (\log |z|)^{\frac{1}{q}} - \varepsilon.$$

Define

$$h^n_{\underline{s}}(t) := g_{s_1} \circ g_{s_2} \circ \cdots \circ g_{s_n}(F^n(t)) \in R_{s_1},$$

where

$$F(z) := \sum |a_n| z^n, \quad f(z) = \sum a_n z^n.$$

Proposition 2

$$|h^n_{\underline{s}}(t)| \geq \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{q}} t - \varepsilon.$$

Proposition 3

$$|g'_{s_i}(z)| \leq \frac{1}{|a|} |z|^{-\frac{1}{\sqrt{2}+\varepsilon}}.$$

Now $h^n_{\underline{s}}(t)$ can be written as follows:

$$h^n_{\underline{s}}(t) = h^1_{\underline{s}}(t) + \sum_{k=1}^{n-1}(h^{k+1}_{\underline{s}}(t) - h^k_{\underline{s}}(t)).$$

Then we have an estimate

$$|h^{k+1}_{\underline{s}}(t) - h^k_{\underline{s}}(t)| \leq \sup_{z\in L}|g'_{s_1}(z)||h^k_{\sigma(\underline{s})}(F(t)) - h^{k-1}_{\sigma(\underline{s})}(F(t))|,$$

where z runs on the line segment L between $h^k_{\sigma(\underline{s})}(F(t))$ and $h^{k-1}_{\sigma(\underline{s})}(F(t))$. By Proposition 2, both points satisfy

$$\begin{aligned} |*| &\geq \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{q}} F(t) - \varepsilon \\ &=: \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{q}} t_1 - \varepsilon. \end{aligned}$$

Since all the components of $f^{-1}(\Gamma)$ in S have the same asymptotics, we have

$$R_{s_i} \cap \{|z| > M\} \subset \left\{ z \,\middle|\, |\arg z| < \frac{\pi}{4q} \right\}.$$

So we have

$$\min_{z\in L}|z| \geq \left(\left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{q}} t_1 - \varepsilon\right)\cos\frac{\pi}{8q}.$$

From the above estimate and Proposition 3, we have

$$\begin{aligned} &\sup_{z\in L}|g'_{s_1}(z)| \\ \leq\ & \frac{1}{|a|}\left(\left((\frac{1}{\sqrt{2}})^{\frac{1}{q}} t_1 - \varepsilon\right)\cos\frac{\pi}{8q}\right)^{-\frac{1}{\sqrt{2}+\varepsilon}}. \end{aligned}$$

Repeating this procedure, we have

$$\begin{aligned} &|h^{n+1}_{\underline{s}}(t) - h^n_{\underline{s}}(t)| \\ \leq\ & \left(\prod_{k=1}^{n}\frac{1}{|a|}\left(\left((\frac{1}{\sqrt{2}})^{\frac{1}{q}} t_k - \varepsilon\right)\cos\frac{\pi}{8q}\right)^{-\frac{1}{\sqrt{2}+\varepsilon}}\right) \\ &\qquad\qquad \times\ |h^1_{\sigma^n(\underline{s})}(F^n(t)) - h^0_{\sigma^n(\underline{s})}(F^n(t))| \\ =\ & \left(\prod_{k=1}^{n} ***\right)|g_{s_{n+1}}(F^{n+1}(t))) - F^n(t)|. \end{aligned}$$

In order to get an estimate for the term $|g_{s_{n+1}}(F^{n+1}(t))) - F^n(t)|$, we n the following propositions:

Proposition 4 $\varphi_{ij} : R_i \to R_j$ is well-defined by the formula

$$f(\varphi_{ij}(z)) = f(z), \quad z \in R_i$$

for z with $|z|$ large enough and satisfies

$$\varphi_{ij}(z) = z + \frac{2(j-i)\pi\sqrt{-1}}{q}\frac{1}{z^{q-1}} + O\Big(\frac{1}{z^{2q-1}}\Big).$$

Proposition 5

$$|g_{s_i}(F(z)) - z| \le \left|\frac{2(s_i - l)\pi\sqrt{-1}}{q} + \varepsilon\right|\frac{1}{|z|^{q-1}},$$

if $z \in R_l$ and $F(z) \in \mathcal{S}$.

By Propostion 5, we have the following estimate:

$$\begin{aligned}
&|h_{\underline{s}}^{n+1}(t) - h_{\underline{s}}^n(t)| \\
&\le \left(\prod_{k=1}^{n}\frac{1}{|a|}\left(\left(\Big(\frac{1}{\sqrt{2}}\Big)^{\frac{1}{q}}t_k - \varepsilon\right)\cos\frac{\pi}{8q}\right)^{-\frac{1}{\sqrt{2}+\varepsilon}}\right) \times \left(\frac{2s_n\pi\sqrt{-1}}{q} + \varepsilon\right)\frac{1}{t_n^{q-1}}
\end{aligned}$$

where $t_k := F^k(t)$. Hence we have

$$g_{\underline{s}}^n(t) \to {}^{\exists}g_{\underline{s}}(t)$$

locally uniformly for $t \ge {}^{\exists}t_0(\ge x)$.

5 Outline of proof of the Main Theorem

Now the proof of (1) and (2) are trivial.

(3) Since $f(h_{\underline{s}}^n(t)) = h_{\sigma(\underline{s})}^{n-1}(F(t))$, we have

$$f(h_{\underline{s}}(t)) = h_{\sigma(\underline{s})}(F(t))$$

by taking a limit. In general we have

$$f^n(h_{\underline{s}}(t)) = h_{\sigma^n(\underline{s})}(F^n(t)) > \Big(\frac{1}{\sqrt{2}}\Big)^{\frac{1}{q}}F^n(t) - \varepsilon.$$

Hence we have,

$$f^n(h_{\underline{s}}(t)) \to \infty \ (n \to \infty).$$

Since it is well known that functions of finite type can have neither Baker domains nor wandering domains, this implies that

$$g_{\underline{s}}(t) \in J(f).$$

(4) Since

$$|h^n_{\underline{s}}(t)| \geq \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{q}} t - \varepsilon$$

from Proposition 2, we have

$$|h_{\underline{s}}(t)| \geq \left(\frac{1}{\sqrt{2}}\right)^{\frac{1}{q}} t - \varepsilon.$$

Therefore

$$\lim_{t \to \infty} h_{\underline{s}}(t) = \infty.$$

(5) Suppose that $h_{\underline{s}}(t)$ is not injective. Then

$$h_{\underline{s}}(t_1) = h_{\underline{s}}(t_2)$$

for some $t_1 < t_2$. Hence we have

$$f^n(h_{\underline{s}}(t_1)) = f^n(h_{\underline{s}}(t_2)),$$

that is,

$$h_{\sigma^n(\underline{s})}(F^n(t_1)) = h_{\sigma^n(\underline{s})}(F^n(t_2)).$$

On the other hand, it turns out that

$$|h_{\sigma^n(\underline{s})}(F^n(t_k)) - F^n(t_k)| < \left|\frac{2s_n\pi\sqrt{-1}}{q} + \varepsilon\right| \frac{1}{|F^n(t_k)|^{q-1}}$$

for $k = 1,\ 2$. Then $g_{\sigma^n(\underline{s})}(F^n(t_k)) - F^n(t_k)$ are bounded for $k = 1,\ 2$ and hence

$$\begin{aligned} & (g_{\sigma^n(\underline{s})}(F^n(t_1)) - F^n(t_1)) - (g_{\sigma^n(\underline{s})}(F^n(t_2)) - F^n(t_2)) \\ = \ & F^n(t_2) - F^n(t_1) \end{aligned}$$

is also bounded, which is impossible.

References

[1] D. Schleicher and J. Zimmer, Dynamic Rays for Exponential Maps, *Preprint SUNY Stony Brook, 1999/9.*

[2] M. Taniguchi, Maskit surgery of entire functions, *Preprint.*